Power Series:

A **power series centered at 0** (or **power series about 0**) in the variable x is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

where the c_n's are constants called the **coefficients** of the series. More generally, a **power series centered at a** (or **power series about a**) is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots.$$

Example 1. Determine the values of x for which each of the following power series converges.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Let's try the ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}$$

$$= \lim_{n\to\infty} \frac{x}{n} = 0$$ + so the series converges for all x.

Note: We will show $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-3)^n}{n}$ $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$= \lim_{n \to \infty} \frac{|x-3|^{n+1}}{n+1} \cdot \frac{n}{|x-3|}$

$= \lim_{n \to \infty} |x-3| \cdot \frac{n}{n+1} = |x-3|$. The series converges for $|x-3| < 1$, $2 < x < 4$. At $x = 2$, the series becomes $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{n}$

$= -\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. At $x = 4$, it is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ which converges. So the series converges for $2 < x \le 4$.

Radius of Convergence:

Theorem: The Power Series Convergence Theorem

If a power series $\sum c_n x^n$ converges for some value $x = b \neq 0$, then it converges for all x such that $|x| < |b|$. On the other hand, if a power series $\sum c_n x^n$ diverges for some value $x = d \neq 0$, then it diverges for all x such that $|x| > |d|$.

Example 2. Determine the radius and interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(11x)^n}{n^4}$.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{11^{n+1}|x|^{n+1}}{(n+1)^4} \cdot \frac{n^4}{11^n |x|^n}$$

$$= \lim_{n\to\infty} 11 \cdot |x| \cdot \frac{n^4}{(n+1)^4} = 11 \cdot |x|.$$ The

series converges for $11 \cdot |x| < 1$ or for $|x| < \frac{1}{11}$. Radius of convergence is $\frac{1}{11}$.

when $x = -\frac{1}{11}$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ converges.

when $x = \frac{1}{11}$, $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges.

The interval of convergence is

$|x| \leq \frac{1}{11}$ or $-\frac{1}{11} \leq x \leq \frac{1}{11}$.

Example 3. Determine the interval of convergence and the limiting function of the series $\sum_{n=0}^{\infty}\left(\frac{x-8}{7}\right)^n$.

Consider $\sum_{n=0}^{\infty}\left|\frac{x-8}{7}\right|^n$. This is a geometric series with $r=\left|\frac{x-8}{7}\right|$.

It converges when $r<1$.

$\left|\frac{x-8}{7}\right|<1$ when $|x-8|<7$. So it converges for $1<x<15$ and the sum is $\frac{a}{1-r}=\frac{1}{1-\left(\frac{x-8}{7}\right)}$

$$=\frac{1}{\frac{7-(x-8)}{7}}=\frac{7}{15-x}.$$

Example 4. Find the power series representation of the function given by $f(x)=\dfrac{2}{1-3x}$ centered at $x=0$. Find the radius of convergence.

Recall that $a + ar + ar^2 + \cdots$

$= \sum_{n=0}^{\infty} ar^n$ is a geometric series

& converges for $-1 < r < 1$.

Let $a = 2$ and $r = 3x$. So

$$f(x) = \frac{2}{1-3x} = \sum_{n=0}^{\infty} 2(3x)^n = \sum_{n=0}^{\infty} 2 \cdot 3^n x^n$$

and it converges for $-1 < 3x < 1$,

$-\frac{1}{3} < x < \frac{1}{3}$.